What's in a Like

A student teacher and her supervisor develop a lesson to connect students' knowledge

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This addition problem was written on the whiteboard:

$$\begin{array}{r} 4x^2 + 5x + 3 \\ +\ 3x^2 + 4x + 6 \\ \hline \end{array}$$

Teacher: Look, x^2 is a double cheeseburger, and x is a cheeseburger. So how many double cheeseburgers do you have?
Students: Four.
Teacher: How many double cheeseburgers do you add?
Students: Three.
Teacher: How many double cheeseburgers do you have in all?
Students: Seven.
Teacher: Great! Get it?

In introductory algebra and later mathematics courses, students seem to struggle with the concepts of like terms and combining like terms with algebraic expressions. These ideas appear to be unfamiliar to students and do not seem to relate to any mathematics that they have done in the past. In an attempt to link with something that the students do

Terms: Name?

of place value to operations with polynomials.

know, teachers may resort to pseudoexplanations or contexts that are not mathematical and that frequently are incorrect concretizations or contextualizations. These concretizations—for example, using different fruits, animals, and foods—incorrectly interpret part-part-whole structures of certain addition problems and do not address the idea that an algebraic expression may contain variable quantities. In the classroom example above, we can certainly join single cheeseburgers (part) and double cheeseburgers (part) in one collection (whole) of cheeseburgers for the given context because the expressions x and x^2 are used to identify names of an object and not to identify variable quantities.

Although using letters to assign attributes of objects or the objects themselves may, at some point, lead some students to combine like terms in limited circumstances, they will be misunderstanding what they are doing. This approach is not linked with any mathematical structure from their prior knowledge; rather, a pseudomathematical explanation is given (Ma 1999). We are not advocating against using context. Proper use of context has been shown to be a fruitful bridge between arithmetic and algebra (Tabach and Friedlander 2008). However, the context must help students develop structure sense (Linchevski and Livneh 1999; Linchevski and Herscovics 1996).

Developing structure sense is given an important place in both NCTM's Process Standards (NCTM 2000) and the Common Core Standards for School Mathematics, particularly the Standards for Mathematical Practices (SMPs) (CCSSI 2010). In this article, we share the collaborative attempt of a student teacher and her supervisor to support the teaching and learning of the concept of combining like terms in algebra with the structural understanding of arithmetic. This is one of several approaches to teaching algebra, of course; for a comprehensive treatise of different perspectives, we recommend Chazan (2000).

BUILDING ON WHAT WE KNOW

The Common Core State Standards for Mathematics (CCSSI 2010) explicitly point out in many places, especially in the K–8 standards, that the learning of new structures should be presented as an extension of fundamental structures already taught. For example, understanding of fractions and rational numbers is built on the structure of whole numbers and integers, so that consistency with operations of numbers is preserved. Further, the Standard for Mathematical

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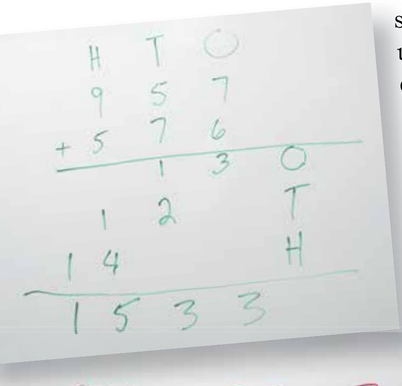
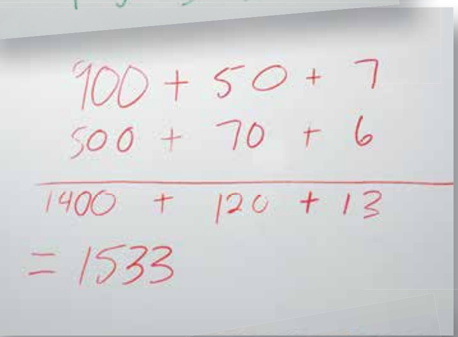
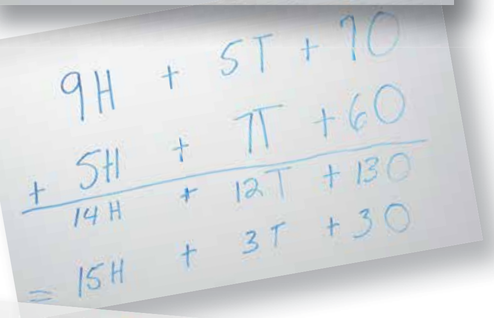
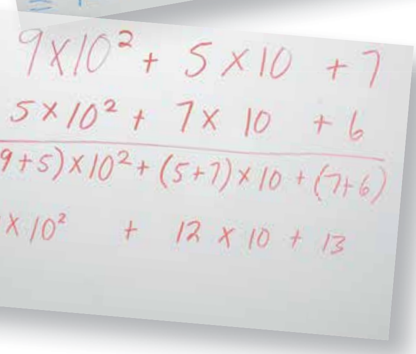
Practice 7—"Look for and make use of structure" (CCSSI 2010, p. 8)—is important at all grade levels within the Common Core. Secondary school teachers of mathematics need to look for and make use of structures in the K–8 standards that can be extended to more generalized ideas in algebra.









We will present one example of how a student teacher, coauthor Boyajian, and her university supervisor, coauthor de Groot, considered this idea in teaching an introductory lesson on polynomial expressions and operations with polynomial expressions. We will present the phases of the lesson that we constructed together during the prelesson discussion, share how this lesson played out in class, and reveal parts of the student teacher's reflection to represent her thinking and learning in hindsight.

In her postobservation reflection of this lesson, the student teacher explains her initial concern, expressed to her supervisor, with regard to how students were learning about like terms in her Algebra 1 class:

> I originally wanted to open my lesson on addition and subtraction of polynomials by referring back to solving systems of linear equations. I wanted to stress that when using the various solution methods, one adds or subtracts like terms. During my lesson I was going to represent the different variables in the examples as different fruits. I knew this was not mathematically correct, but this is what my cooperating teacher had always done in the past, so I did not want to override her way of teaching addition and subtraction of polynomials. But it just did not feel right.

To address this concern, we began by finding past experiences in which Boyajian's students had shown competence. We determined that her students were comfortable and skilled with the base-10 place-value system, through which they had learned the fundamental structure of numeric polynomial expressions and how to add and subtract them—that is, to combine like terms. We can consider a multidigit number—for example, 957—as a numeric polynomial expression by expanding it as $9 \cdot 10^2 + 5 \cdot 10 + 7$. Boyajian articulated this thinking in her postlesson reflection:

> Based on my misgivings, I decided to have a discussion with my supervisor about what addition and subtraction of polynomials actually meant. We talked about how students have been adding and subtracting polynomials since second grade by thinking of numbers as polynomials—[that is,] as objects that have "many names." This idea of objects with more than one name became a big aha! moment for me, and I decided with my supervisor to completely change the launch phase of the lesson to establish this context for polynomials and operations with polynomials.

The context for this work was established by beginning with adding multidigit numbers in an attempt to develop the idea that these can be written as numeric polynomial expressions providing an algebraic protostructure for addition and subtraction. We can write a multidigit number in polynomial form by writing it in expanded scientific notation (see the example of 957 above). Here we will refer to multidigit numbers as numeric polynomials. Each step of the instructional process that follows was recorded on the cue cards that we generated in preparation.

Step 1: Establishing Competence

The student teacher began by writing the three-digit number 576 on the board and asking the students what the value of each digit was. She then emphasized that students had identified the value by naming it, that each digit has its own name. She then asked the students to show how they would add 957 and 576. The students said that they needed to line up the digits and then add column by column; they used the traditional addition algorithm to complete the task. The student teacher then asked why students did not add the two sevens together. They responded that the sevens were not in the same column and that you can add only digits that are in the same column. Next, the student teacher shifted the language to adding digits that have the same name: ones with ones, tens with tens, and hundreds with hundreds. She represented this shift on the board as shown in **figure 1**.

The student teacher guided the students to see that 6 ones and 7 ones added to 13 ones; that 7 tens and 5 tens added to 12 tens; and that 5 hundreds and 9 hundreds added to 14 hundreds. Note that each partial sum moves up one place value as we move to each successive column; that it does not matter in what order we find the three partial sums; and that we add digits that have the same name. This was a novel point of view for the students, but they found it easier to work with than the traditional method.

Step 2: Representing Multidigit Integers as Polynomial Expressions Using Expanded Notation

A multidigit number does not yet look like a numeric polynomial expression. We found that some important misconceptions about polynomial expressions can be cleared up using the familiarity with the base-10 place-value number system. In the postlesson reflection, Boyajian wrote:

> Many of the students were becoming involved with the lesson and started asking questions. We determined that polynomials are expressions with terms that have different names. Then one student asked me, "What about 444?" The student appeared to be thinking that since there are all 4s in the expression, they cannot have different names. I then replied, "Well, in the number 444, we have 4 hundreds, 4 tens, and 4 ones. Do you see the difference? That was a great question!"

This question nicely led to the next step of writing the addends in expanded notation. The addition problem was now presented as shown in **figure 2**.

Several students were able to connect the work in step 1 with that in step 2. They realized that 12 tens is equivalent to 120 and that 14 hundreds is equivalent to 1400. They also began to articulate during the lesson that step 2 looked more like their prior work on solving systems of two linear equations in two variables, where they learned to add and subtract like terms using the elimination method. Although adding polynomial expressions is not the same as adding linear combinations of equations, the concept of like terms is essential in both structures. Students began to intuit this idea and began to realize that naming is an important idea that occurs with both numbers and variables.

		H	T	O	
		9	5	7	
+		5	7	6	
			1	3	O
		1	2		T
	1	4			H
	1	5	3	3	

Fig. 1 Student competence with past experience establishes a foundation.

$$\begin{array}{rrrrrr} & 900 & + & 50 & + & 7 \\ + & 500 & + & 70 & + & 6 \\ \hline & 1400 & + & 120 & + & 13 \\ = & 1533 & & & & \end{array}$$

Fig. 2 Expanded notation is used to calculate a sum.

$$\begin{array}{rrrrrr} & 9H & + & 5T & + & 7O \\ + & 5H & + & 7T & + & 6O \\ \hline & 14H & + & 12T & + & 13O \\ = & 15H & + & 3T & + & 3O \end{array}$$

Fig. 3 Digits are named according to place value.

Step 3: Linking Number as Polynomial Expression with Place Value

In step 3, we attempted to connect the first two steps by explicitly naming each digit according to its place value. At this stage, we must be careful not to interpret the names as variables. Thus, 9H represents nine units of hundreds, 5T represents five units of tens, and 7O represents seven units of ones. Note that we exchanged 10 ones for 1 ten and then 10 tens for 1 hundred (see **fig. 3**) after we did the addition according to the algebraic structure of adding same named quantities. Exchanging can also be thought of as renaming in a different unit. In algebra, we do not do such renaming in a different unit because the base is variable.

Several students started to notice that the H, T, and O seemed to function just like variables, and some connected the distributive property to this process of addition. Connecting polynomials with place value clarifies the transition for students to visualize how the structure of a numeric polynomial is similar to that of an algebraic polynomial written with variables. The student teacher was careful to make sure that at this stage the letters H, T, and O represented names of units with fixed values. This thinking led to the next step.

Step 4: Connecting Scientific Notation with Numeric Polynomial Expressions, Place Value, and the Distributive Property

We wanted to help students see the role of the distributive property more explicitly and then transition to helping them see that algebraic polynomial addition is fundamentally the same in structure as numeric polynomial addition. To do so, we decided to rewrite the expanded notation of step 2 in scientific notation. This was a crucial but not a simple step in the development of the lesson. Boyajian described this process in her postlesson reflection:

> "Are you familiar with scientific notation?" was the transition question I asked my students when shifting from step 3 to step 4. The students replied yes. I then told the class I could

$$\begin{array}{ccccccc} & 9\times 10^2 & + & 5\times 10 & + & 7 \\ + & 5\times 10^2 & + & 7\times 10 & + & 6 \\ \hline & (9+5)\times 10^2 & + & (5+7)\times 10 & + & (7+6) \\ = & 14\times 10^2 & + & 12\times 10 & + & 13 \end{array}$$

Fig. 4 Scientific notation and the distributive property support the transition to work with algebraic polynomials.

$$\begin{array}{ccccccc} & 3\cdot 12^2 & + & 7\cdot 12 & + & 6 \\ + & 6\cdot 12^2 & + & 5\cdot 12 & + & 7 \\ \hline & (3+6)\cdot 12^2 & + & (7+5)\cdot 12 & + & (6+7) \\ = & 9\cdot 12^2 & + & 12\cdot 12 & + & 13 \end{array}$$

Fig. 5 Structure allows for generalization.

> rewrite the numeric polynomials using scientific notation, because this uses 10 as the base and connects directly to the base-10 notation system. The students looked momentarily panicked until I informed the class that we would use it to discover something important.
>
> I explained to the students that if 900 = 9(100) and 100 = (10^2), then 900 = 9(10^2). We concluded that the digit 9 in the hundreds place has a value of 9(10^2). Likewise, the digit 5 in the tens place has a value of 5(10^1), and the digit 7 in the ones place has a value of 7(10^0). The students saw that the place of the digit was represented by the exponent of the base. Because of the consistency of adding terms in the same column, the students began seeing the pattern and relationship between a digit's value according to its place and like terms.
>
> After I set up the addends with scientific notation, the class began to see that we used the distributive property to add quantities that have the same name, that are in the same place, and that are like terms: 9(10^2) + 5(10^2) = (9 + 5)(10^2) = 14(10^2). It was in this moment of the lesson that I started to really get it myself, and I didn't need my notes anymore. My language had changed to mathematical language, and so had my students.' No more cheeseburgers for me!

During this step, the student teacher and her students had changed their conceptions. At this point, she put the lesson notes on her desk. New thinking had become structurally clear and had become a strong foundation for moving forward.

During step 3, we did include an exchange (a renaming) of 10 tens for one hundred; we did not do this in step 4. There we transitioned to the algebraic way of adding like terms using the distributive property and did not exchange anymore. Algebraically, base 10 is just one of many possible systems. In another base, the coefficients 14, 12, and 13 will be decomposed differently (see **fig. 4**). Therefore, algebraically we cannot exchange because the base is variable. Here is where adding algebraic polynomials departs from adding numeric polynomials. This is why step 4 is such a crucial transition.

Step 5: Generalizing to Addition with Algebraic Polynomial Expressions

Using the scientific notation allowed us to abandon the specific case of the base-10 system and work toward the idea that in an algebraic polynomial the base of the system can be thought of as a variable. **Figure 5** shows one of the many examples that were used—in this case, a base-12 example.

It took a good number of examples for students to abstract the structure to that of an algebraic polynomial. In the days following this lesson, the students looked at adding polynomials with a stronger structural perspective that was connected and deeply mathematical in nature. Note that the numeric polynomial sum in **figure 5** can be renamed as $10 \cdot 12^2 + 1 \cdot 12 + 1$. For algebraic polynomial expressions, we cannot accomplish such renaming in general because the base is variable, according to this perspective on such expressions. To rename a multiple of x as x^2, we will need x groups of x.

In this article, we have considered only situations in which the terms are added. We have not addressed combining like terms of expressions with mixed signs, something that will need to be placed in the context of adding and subtracting integers. For example, we can expand 376 as $4 \cdot 10^2 - 2 \cdot 10 - 4$ and then find its sum or difference from another numeric polynomial.

A POLYNOMIAL BY MANY NAMES

In this article, we have tried to accomplish two things. First, we wanted to share our collaboration as student teacher and university supervisor during a student-teaching experience. Second, we wanted to share this experience in the context of linking important algebra concepts with fundamental structures of K–12 mathematics. In the course of this and subsequent lessons, both the students' and the student teacher's mathematical language changed, revealing insight into the structural properties of polynomial addition. In addition, the importance of the distributive property was amplified in this process. Translating *polynomial* to mean "many names" was not just a semantic device but a discovery that naming is our human way of distinguishing one place value from another and one term from another. We hope that you find equal promise in these ideas.

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